Remainder formula- Quant Study Notes for Competitive Exams

Today we will be covering following topics in this blog namely, Remainder Formula, Fermat Little's Theorem, Wilson's Theorem, Finding Last two digits when divided by a number with large exponent. So let's learn then.

Remainder Formula

Let's just start from where we had left in the earlier section. We can find whether any number is divisible by 9, 99, 999, 11, 101, 1001, etc. or not. There were two cases:

1. The number is fully divided by these numbers.
2. The number is not fully divided by these numbers.

In the second case, after doing our calculation, we will be left with a number which is smaller than the dividend (if it is not smaller, we will do that process again until we will find a number smaller than the dividend), that number will be the remainder.

E.g. Find the remainder when 83421579 is divided by 1001. Sol: (Tripling from right side):- (579 + 83) – 421 = 241, Hence, the remainder will be 241.

E.g. Find the remainder when 212121......[120 digits] is divided by 99. Sol: There are 120 digits, so there are 60 pairs of 21. So, the sum of 60 pairs will be, 21 × 60 = 1260 Now, we will apply same procedure for 1260 (Pairing from right side):- 60 + 12 = 72. Hence, the remainder will be 72.

E.g. Find the remainder when 324516.....[600 digits] is divided by 1001. Sol: There are 600 digits. It means there are 200 triplets in which 100 triplets are of 516 and 100 are of 324. We have to take the difference of the sum of all the triplets at even position from right side (i.e. 324 × 100) and the sum of all the triplets at odd position (i.e. 516 × 100) ⇒ (516 – 324) × 100 = 19200 We have to do this procedure again, (Tripling from right side): 200 – 19 = 181. Hence, 181 will be the remainder. Now, we know that 1001 = 7 × 11 × 13. So, we just have to divide the remainder of the number, when divided by 1001, by 7. And the remainder will be the final remainder.

Like, in example 7, the remainder is 241. If we divide 241 with 7, the remainder will be 3. So, when 83421579 is divided by 7, the remainder will be 3. Same for 13. Like, in some pevious example, the remainder is 181. If we divide 181 with 13, the remainder will be 12. So, when 324516 ....[600 digits] is divided by 13, the remainder will be 12.

► Now, we know that, like, let a number (say N), when divided by 9, gives the remainder of 5. So, if the same number is divided by 3 will give the remainder of 2. It is just because, we can write **N as: N = 9k + 5**

Here, since 3 is a factor of 9, 9k will be divisible by 3. And for 5, when divided by 3, the remainder will be 2. And, if N is divided by 99 and the remainder is 5. Then the remainder will be 5 if N is divided by 11 or 9. Because, 11 and 9 are factors of 99 and 5 is less than 9. Let's just check out some properties of remainder: □

For a number n^m, where n = pq + r (where q is quotient, r is remainder and p is dividend), when n^m is divided by p, the remainder will be the same as the remainder obtained when r^m is divided by p. (We can prove this by taking power m on both sides but there is no need.)

The remainder r and the number (r – p) are equivalent, but since we find the positive remainder, we should take the positive value. Now, with some examples, let's just try to find the remainder of other numbers.

E.g. Find the remainder when 5^{30} is divided by 4. Sol: Since, if we divide 5 with 4, the remainder will be 1. And 1^{30} is always 1. So, the remainder will be 1.

E.g. Find the remainder when 8^{37} is divided by 9. Sol: $8^{37}/9 = (-1)^{37}/9 = -1/9 = (9 - 1)/9 = 8/9$ So, the remainder will be 8.

E.g. Find the remainder when 7^{57} is divided by 9. Sol: $7^{57}/9 = (-2)^{57}/9 = -(2)^{57}/9$ (Since the power is odd, the term will be negative) $\Rightarrow -(2^3)^{19}/9 = -(8)^{19}/9 = -(-1)^{19}/9 = 1$ So, the remainder will be 1.

E.g. Find the remainder when 32^{33} is divided by 7. Sol: $32^{33}/7 = 4^{33}/7 = (4^3)^{11}/7 = 64^{11}/7 = 1^{11}/7 = 1/7$ So, the remainder will be 1.

If $p = n \times s$, so for the remainder of n^m, when divided by p, we can find the remainder of the term $n^{(m-1)}$ (or accordingly, as per the requirement of the question), when divided by s. and later multiply the remainder with n.

E.g. Find the remainder when 7^{71} is divided by 35. Sol: $7^{71}/35$. Since, $35 = 7 \times 5$, one 7 will be cancelled by 35. So, it becomes, $7^{70}/5 = (7^2)^{35}/5 = (4^9)^{35}/5 = (-1)^{35}/5 = -1/5 = (5 - 1)/5 = 4/5 = 28/35$ So, the remainder will be 28.

E.g. Find the remainder when 21^{73} is divided by 16. Sol: $21^{73}/16 = 5^{73}/16 = (5^2)^{36}(5)/16 = (25)^{36}(5)/16 = 9^{36}(5)/16$ $(9^2)^{18}(5)/16 = (81)^{18}(5)/16 = (1)^{18}(5)/16 = 5/16$ So, the remainder will be 5.

We can find the remainders when the dividends are small or have factors, but what if the dividends are big prime numbers?

Fermat Little's Theorem

Remainder $[N^{P-1}/P] = 1$ (where, P is a prime number and $N < P$) Or, Remainder $[a^{P-1}/P] = 1$ [where, P is a prime number and HCF (a, P) = 1]

E.g. Find the remainder when 3^{83} is divided by 41. Sol: since 41 is a prime number so 3^{40} when divided by 41, the remainder will be 1. So, $3^3/41 = 27/41$ So, the remainder will be 27.

E.g. Find the remainder when 87^{82} is divided by 17. Sol: since 17 is a prime and HCF (87, 17) = 1. So, $87^{16}/17 = 1$. And, $(87^{16})^5/7 = 1$. So, we are left with, $87^2/17 = 2^2/17 = 4/17$ (Because $87 = 17 \times 5 + 2$) So, the remainder will be 4.

(We can do this directly by taking $87^{16}/17 = 2^{16}/17$ and then by applying Fermat's little theorem)

Wilson's Theorem

- Remainder [(P – 1)!/P] = -1 or (P – 1)
- Remainder [(P – 2)!/P] = 1
- Remainder [(P – 3)!/P] = (P – 1)/2 (Where P is a prime number)

E.g. Find the remainder when $(14!)^{38}$ is divided by 17. Sol: since, 14 = 17 – 3, we can apply the third Wilson theorem. ⇒Remainder (14!/17) = (17 – 1)/2 = 8 So, $8^{38}/17 = (8^{16})^{2}(8)^{6}/17 = 8^{6}/17$ = 4 Hence, the remainder will be 4.

E.g. Find the remainder when 21! Is divided by 19. Sol: Since 21! = 21 × 20 × 19 × … × 1, 19 will be cancelled out with 19. So, the remainder will be 0.

E.g. Find the remainder when 19! Is divided by 23. Sol: Since we can't apply any Wilson theorem, we can assume that, Remainder [19!/23] = k Now, Remainder [21!/23] = 1 ⇒1Remainder [(21 × 20 × 19!)/23] = 1 ⇒ Remainder [(-2)(-3)(k)/23] = 1 ⇒Remainder [6k/23] = 1 For, k = 4, the remainder will be 1. So, the remainder will be 4.

►To Find the remainder of n^m, when divided by p, where p = x × y, we can separately find the remainder of nm, when divided by x and y, and then we will join them by making an LCM model and find the least value. So, n^m = k × p + (least value). And that least value will be our remainder. Let's understand it better by taking an example.

E.g. Find the remainder when 2^{1024} is divided by 77. Sol: We know that 77 = 11 × 7. Since both are prime numbers and 2 is smaller than both of them, we can use Fermat's little theorem. ⇒ $2^{1024}/7 = (2^{6})^{170}(2)^{4}/7 = 24/7 = 16/7 = 2/7$ So, 2^{1024} = 7a + 2 …..(i) Also, $2^{1024}/11 = (2^{10})^{102}(2)^{4}/11 = 2^{4}/11 = 16/11 = 5/11$ So, 2^{1024} = 11b + 5 …..(ii)

Now, we have to find the least value of a and b. We put both the equations together, ⇒11b + 5 = 7a + 2 ⇒ 7a = 11b + 3 ⇒ 7a = 7b + (4b + 3) It means, 4b + 3 = 0 or 7 or 14 or 21 or etc. For b = 1, 4b + 3 will be 7. So, b = 1 (least value). So, the least value will be, ⇒ 11 × 1 + 5 = 16. Hence, we can represent it as, ⇒2^{1024} = 77n + 16. So, the remainder will be 16.

Rules for numbers in the form of $a^n – b^n$

- It is always divisible by (a – b)
- When n is even, it is also divisible by (a + b)

Rules for numbers in the form of $a^n + b^n$

- When n is odd, it is divisible by (a + b)
- We can't say anything when n is even.

Last two digits

We also saw in exams, the questions like, what will be the last two digits of 21^{137} or what will be the remainder if 21^{137} is divided by 100. See, we can clearly say that since the unit digit is 1, the last digit of the answer will be 1. But we are not sure about the second last digit. So, let's check the last two digits of the power of the number containing unit digit in base as:

- 0 → 00, For any number ending with 0, like 70126, 2035, 30001235 , 764501235, the last two digits will always be 00.
- 5 → It has two cases, i) The power and the second last digit of base is odd → 75 ii) Otherwise →25

E.g. Find the remainder when $(275)^{123}$ is divided by 100. Sol: Here, we have to calculate the last two digits of the number. Since the second last digit (i.e. 7) and the power (i.e. 123) both are odd, so the last two digits will be 75. So, the remainder will be 75.

E.g. Find the last two digits of $(385)^{167}$. Sol: Since, the second last digit of base is even, so the last two digits will be 25.

- 1 → Last digit is 1 And the second last digit is the last digit of the product of the second last digit of base and last digit of power. i.e. for number like (xy1)abc , Second last digit = last digit of (y × c)
- 3 → It has two cases,3, 7, 9 → We know that every fourth power of 3 and 7 is 1 and every second power of 9 is 1. So, we have to convert the unit digit into one, then apply the trick given above.

E.g. Find the last two digits of $(83)^{382}$. Sol: (I'll mention last two digits only, so be aware) ⇒ $(83)^{382} = (83^4)^{95} (83^2)$ Now, $83^4 = (83^2)(83^2)$ → (89)2 → 21 ⇒ (21)95 (83)2 = (01) (89) = 89 So, the last two digits will be 89.

E.g. Find the last two digits of $(47)^{97}$. Sol: $(47)^{97} = (47^4)^{24} (47)$ Now, $47^4 = (47^2)^2 \Rightarrow (09)^2 = 81 \Rightarrow (81)^{24} (47) \rightarrow 21 \times 47 = 87$ So, the last two digits will be 87.

- 2 ⇒It has two cases, 2 ⇒ There are only 4 things to remember (apart from all values from 2^1 to 2^{10}, which we have to learn) Last two digits of 2^{10} = 24. Last two digits of 24^{odd} = 24, Last two digits of 24^{even} = 76, Last two digits of 76^{any} = 76.
- 8⇒ It has two cases,4, 6, 8 ⇒ We have to split the number as a multiple of 2. Then we will apply all the previous tricks.

E.g. Find the last two digits of $(98)^{84}$. Sol: $98^{84} = 2^{84} \times 49^{84} \Rightarrow (2^{10})^8 (2^4) \times (49^2)^{42} \Rightarrow 76 \times 16 \times (01)^{42} \rightarrow 16 \times 01 = 16$ So, the last two digits will be 16.

E.g. Find the remainder when $(158)^{432}$ is divided by 100. Sol: $(158)^{432} \Rightarrow (58)^{432} = 2^{432} \times 29^{432} = (2^{10})^{43} \times 2^2 \times (29^2)^{216} \Rightarrow 24 \times 4 \times (29^2)^{216} \Rightarrow 96 \times (41)^{216} \Rightarrow 96 \times 41 \Rightarrow 36$ So, the remainder will be 36.

[Note: Last two digits of $k^2 = (50 - k)^2 = (100 - k)^2 = (100 + k)^2$]

So, that's all we have in Number System. With some seriousness and some practice, you will master this topic. Stay tuned for more.